

Expansionfree Fluid Evolution and Skripkin Model in $f(R)$ Theory

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Abstract

We consider the modified $f(R)$ theory of gravity whose higher order curvature terms are interpreted as a gravitational fluid or dark source. The gravitational collapse of a spherically symmetric star, made up of locally anisotropic viscous fluid, is studied under the general influence of the curvature fluid. Dynamical equations and junction conditions are modified in the context of $f(R)$ dark energy and by taking into account the expansionfree evolution of the self-gravitating fluid. As a particular example, the Skripkin model is investigated which corresponds to isotropic pressure with constant energy density. The results are compared with corresponding results in General Relativity.

Keywords: $f(R)$ theory; Viscous anisotropic fluid; Skripkin model.

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1 Introduction

Modified $f(R)$ theory of gravity constitutes an important development in modern cosmology and theoretical physics. One of the main motivations of this theory is that it may lead to some interesting results about dark

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energy (DE). The other motivation comes from the fact that every unification of fundamental interaction exhibits effective actions containing higher order terms in the curvature invariants. This strategy was adopted in the study of quantum field theory in curved spacetimes [1] as well as in the Lagrangian of string and Kaluza-Klein theories [2].

Higher order terms always give an even number as an order of the field equations. For example, R^2 term produces fourth order field equations [3], term $R\Box R$ (where $\Box \equiv \nabla^\mu \nabla_\mu$) gives sixth order field equations [4, 5], similarly, $R\Box^2 R$ yields eighth order field equations [6] and so on. Using conformal transformation, the term with second derivative corresponds to a scalar field. For instance, fourth order gravitational theory corresponds to Einstein theory with one scalar field, sixth order gravity corresponds to Einstein gravity with two scalar fields, *etc.* [4, 7]. In this context, it is easy to show that $f(R)$ gravity is equivalent to scalar tensor theory as well as to General Relativity (GR) with an ideal fluid [8].

Let us now show that how $f(R)$ gravity can be related to the problem of DE by a straightforward argument. When the Einstein-Hilbert (EH) gravitational action in GR,

$$S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \quad (1.1)$$

is written in the modified form as follows

$$S_{f(R)} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R), \quad (1.2)$$

the addition of a non-linear function of the Ricci scalar demonstrates to cause acceleration for a wide variety of $f(R)$ function, e.g., [9]-[18]. Variation of $f(R)$ action with respect to the metric tensor leads to the following fourth order partial differential equations

$$F(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} - \nabla_\alpha \nabla_\beta F(R) + g_{\alpha\beta}\Box F(R) = \kappa T_{\alpha\beta}, \quad (\alpha, \beta = 0, 1, 2, 3), \quad (1.3)$$

where $F(R) \equiv df(R)/dR$. Writing this equation in the form of Einstein tensor, it follows that [19]

$$G_{\alpha\beta} = \frac{\kappa}{F}(T_{\alpha\beta}^m + T_{\alpha\beta}^{(D)}), \quad (1.4)$$

where

$$T_{\alpha\beta}^{(D)} = \frac{1}{\kappa} \left[\frac{f(R) + RF(R)}{2} g_{\alpha\beta} + \nabla_\alpha \nabla_\beta F(R) - g_{\alpha\beta} \square F(R) \right]. \quad (1.5)$$

It is clear from Eq.(1.4) that curvature stress-energy tensor $T_{\alpha\beta}^{(D)}$ formally plays the role of a source in the field equations and its effect is the same as that of an effective fluid of purely geometrical origin. In fact, this scheme provides all the ingredients needed to tackle the dark side of the universe. Thus curvature fluid can play the role of both dark matter and DE. The "Dark source" term is not restricted to hold the usual energy conditions. Therefore, $f(R)$ theory may be used to explain the effects of DE on cosmological and gravitational phenomena of the universe.

The importance of gravitational collapse lies at the center of structure formation in the universe. A starting smooth arrangement of matter will eventually collapse and make the powerful structures such as stellar groups, planets and stars. One can study the gravitational collapse by taking the interior and exterior regions of spacetime. The proper junction conditions help to study the smooth matching of these regions. In Einstein gravity, the pioneer work [20] was carried out on dust collapse by taking static Schwarzschild in the exterior and Friedmann like solution in the interior spacetime.

During fluid evolution, self-gravitating objects may pass through phases of intense dynamical activities. The dynamical equations are used to observe the effects of dissipation over collapsing process. Skripkin [21] studied the central explosion of the spherically symmetric fluid distribution under condition of constant energy density. The evolution of such a fluid yields the formation of a Minkowskian cavity within the fluid distribution centered at the origin. This problem is studied in detail by Herrera *et al.* [22, 23]. It was shown that under Skripkin conditions, the expansion scalar vanishes which requires the existence of a cavity within the fluid. Sharif *et al.* investigated some aspects of gravitational collapse regarding singularity and event horizons in $f(R)$ theory [24] and also, in GR dissipative fluid collapse with and without adding charge [25]-[27]. In a recent paper [28], we have studied the effects of $f(R)$ DE on a dissipative collapse. Some special solutions are also discussed for nondissipative case.

In this work, we investigate that how DE generated by curvature fluid generally affects the dynamics of viscous self-gravitating fluid. In fact, this provides the generalization and extension of our previous work [28]. We

explore the contribution of $f(R)$ DE in developing Skripkin model. The plan of the paper is as follows. In next section **2**, we formulate the field equations in metric $f(R)$ gravity for spherically symmetric distribution of anisotropic fluid. Section **3** is devoted to derive junction conditions and dynamical equations with the inclusion of curvature fluid. In section **4**, fluid evolution is investigated under the assumption of vanishing scalar expansion. Section **5** is devoted to study the Skripkin model in $f(R)$ gravity. The last section **6** concludes the main results of the paper.

2 Fluid Distribution and the Field Equations

We consider a $3D$ hypersurface $\Sigma^{(e)}$, an external boundary of the star, which divides a $4D$ spherically symmetric spacetime into two regions named as interior and exterior spacetimes. For the interior region to $\Sigma^{(e)}$, we take the line element in the form as

$$ds_-^2 = A^2(t, r)dt^2 - B^2(t, r)dr^2 - C^2(t, r)(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.1)$$

For the exterior spacetime to $\Sigma^{(e)}$, we take the Schwarzschild spacetime given by the line element

$$ds_+^2 = (1 - \frac{2m}{r})d\nu^2 + 2drd\nu - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2.2)$$

where m represents the total mass and ν is the retarded time.

In the interior region, we assume a distribution of anisotropic self-gravitating fluid which undergoes dissipation in the form of shear viscosity. The energy-momentum tensor for such a fluid is defined as

$$T_{\alpha\beta} = (\rho + p_\perp)u_\alpha u_\beta - p_\perp g_{\alpha\beta} + (p_r - p_\perp)\chi_\alpha \chi_\beta - 2\eta\sigma_{\alpha\beta}. \quad (2.3)$$

Here, we have ρ as the energy density, p_\perp the tangential pressure, p_r the radial pressure, η the coefficient of shear viscosity, u_α the four-velocity of the fluid and χ_α the unit four-vector along the radial direction. These quantities satisfy the relations

$$u^\alpha u_\alpha = 1, \quad \chi^\alpha \chi_\alpha = -1, \quad \chi^\alpha u_\alpha = 0 \quad (2.4)$$

which are obtained from the following definitions in co-moving coordinates

$$u^\alpha = A^{-1}\delta_0^\alpha, \quad \chi^\alpha = B^{-1}\delta_1^\alpha. \quad (2.5)$$

The shear tensor σ_{ab} is defined by

$$\sigma_{\alpha\beta} = u_{(\alpha;\beta)} - a_{(\alpha}u_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} - u_{\alpha}u_{\beta}), \quad (2.6)$$

where the acceleration a_a and the expansion scalar Θ are given by

$$a_{\alpha} = u_{\alpha;\beta}u^{\beta}, \quad \Theta = u^{\alpha}_{;\alpha}. \quad (2.7)$$

The bulk viscosity does not appear as it can be absorbed in the form of radial and tangential pressures of the self-gravitating fluid. From Eqs.(2.5) and (2.6), we have the following non-vanishing components of the shear tensor

$$\sigma_{11} = -\frac{2}{3}B^2\sigma, \quad \sigma_{22} = \frac{1}{3}C^2\sigma, \quad \sigma_{33} = \sigma_{22}\sin^2\theta, \quad (2.8)$$

where σ is the shear scalar and is given by

$$\sigma = \frac{1}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right). \quad (2.9)$$

From Eqs.(2.5) and (2.7), we have

$$a_1 = -\frac{A'}{A}, \quad a^2 = a^{\alpha}a_{\alpha} = \left(\frac{A'}{AB} \right)^2, \quad \Theta = \frac{1}{A} \left(\frac{\dot{B}}{B} + 2\frac{\dot{C}}{C} \right), \quad (2.10)$$

where dot and prime represent derivative with respect to t and r respectively.

In Einstein frame, the field equations (1.4) for the interior metric become

$$\begin{aligned} & \left(\frac{2\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} - \left(\frac{A}{B} \right)^2 \left[\frac{2C''}{C} + \left(\frac{C'}{C} \right)^2 - \frac{2B'C'}{BC} - \left(\frac{B}{C} \right)^2 \right] \\ &= \frac{8\pi}{F} [\rho A^2 + T_{00}^{(D)}], \end{aligned} \quad (2.11)$$

$$-2 \left(\frac{\dot{C}'}{C} - \frac{\dot{C}A'}{CA} - \frac{\dot{B}C'}{BC} \right) = \frac{8\pi}{F} T_{01}^{(D)}, \quad (2.12)$$

$$- \left(\frac{B}{A} \right)^2 \left[\frac{2\ddot{C}}{C} - \left(\frac{2\dot{A}}{A} - \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} \right] + \left(\frac{2A'}{A} + \frac{C'}{C} \right) \frac{C'}{C} - \left(\frac{B}{C} \right)^2$$

$$= \frac{8\pi}{F}[(p_r + \frac{4}{3}\eta\sigma)B^2 + T_{11}^{(D)}], \quad (2.13)$$

$$\begin{aligned} & - \left(\frac{C}{A}\right)^2 \left[\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{A}}{A} \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \frac{\dot{B}\dot{C}}{BC} \right] + \left(\frac{C}{B}\right)^2 \left[\frac{A''}{A} + \frac{C''}{C} \right. \\ & \left. - \frac{A'B'}{AB} + \left(\frac{A'}{A} - \frac{B'}{B} \right) \frac{C'}{C} \right] = \frac{8\pi}{F}[(p_\perp - \frac{2}{3}\eta\sigma)C^2 + T_{22}^{(D)}]. \end{aligned} \quad (2.14)$$

Here the components of dark fluid are obtained from Eq.(1.5) as follows

$$\begin{aligned} T_{00}^{(D)} &= \frac{A^2}{8\pi} \left[\frac{f + RF}{2} + \frac{F''}{B^2} + \left(\frac{2\dot{C}}{C} - \frac{\dot{B}}{B} \right) \frac{\dot{F}}{A^2} + \left(\frac{2C'}{C} - \frac{B'}{B} \right) \frac{F'}{B^2} \right], \\ T_{01}^{(D)} &= \frac{1}{8\pi} \left(\dot{F}' - \frac{A'}{A} \dot{F} - \frac{\dot{B}}{B} F' \right), \\ T_{11}^{(D)} &= \frac{-B^2}{8\pi} \left[\frac{f + RF}{2} - \frac{\ddot{F}}{A^2} + \left(\frac{\dot{A}}{A} + \frac{2\dot{C}}{C} \right) \frac{\dot{F}}{A^2} + \left(\frac{A'}{A} + \frac{2C'}{C} \right) \frac{F'}{B^2} \right], \\ T_{22}^{(D)} &= \frac{-C^2}{8\pi} \left[\frac{f + RF}{2} - \frac{\ddot{F}}{A^2} + \frac{F''}{B^2} + \left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{\dot{F}}{A^2} \right. \\ & \quad \left. + \left(\frac{A'}{A} - \frac{B'}{B} + \frac{C'}{C} \right) \frac{F'}{B^2} \right]. \end{aligned} \quad (2.15)$$

The Ricci scalar curvature is given by

$$\begin{aligned} R &= 2 \left[\frac{A''}{AB^2} - \frac{\ddot{B}}{A^2B} + \frac{\dot{A}\dot{B}}{A^3B} - \frac{A'B'}{AB^3} + \frac{2\ddot{C}}{CA^2} + \frac{2\dot{A}\dot{C}}{ACB^2} \right. \\ & \quad \left. + \frac{2C''}{CB^2} - \frac{2\dot{C}\dot{B}}{CA^2B} - \frac{2C'B'}{CB^3} - \frac{1}{C^2} - \frac{\dot{C}^2}{A^2C^2} + \frac{C'^2}{B^2C^2} \right]. \end{aligned} \quad (2.16)$$

3 Junction Conditions and the Dynamical Equations

Here, we develop equations that govern the dynamics of dissipative spherically symmetric collapsing process. For this purpose, we use Misner and

Sharp formalism [29]. The mass function is defined by

$$M(t, r) = \frac{C}{2}(1 + g^{\mu\nu}C_{,\mu}C_{,\nu}) = \frac{C}{2} \left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2} \right). \quad (3.17)$$

From the continuity of the first and second differential forms, the matching of the nonadiabatic sphere to the Schwarzschild spacetime on the boundary surface, $\Sigma^{(e)}$, yields the following results

$$M(t, r) \stackrel{\Sigma^{(e)}}{=} m \quad (3.18)$$

and

$$\begin{aligned} 2 \left(\frac{\dot{C}'}{C} - \frac{\dot{C}A'}{CA} - \frac{\dot{B}C'}{BC} \right) \stackrel{\Sigma^{(e)}}{=} -\frac{B}{A} \left[\frac{2\ddot{C}}{C} - \left(\frac{2\dot{A}}{A} - \frac{\dot{C}}{C} \right) \frac{\dot{C}}{C} \right] \\ + \frac{A}{B} \left[\left(\frac{2A'}{A} + \frac{C'}{C} \right) \frac{C'}{C} - \left(\frac{B}{C} \right)^2 \right]. \end{aligned} \quad (3.19)$$

The important reason for generating the junction conditions at the stellar surface is to study the dissipative evolution of the star. Using the field equations (2.12) and (2.13) in Eq.(3.19), we obtain

$$-p_r - \frac{4}{3}\eta\sigma \stackrel{\Sigma^{(e)}}{=} \frac{T_{11}^{(D)}}{B^2} + \frac{T_{01}^{(D)}}{AB}. \quad (3.20)$$

In the next section, we would discuss dynamics with physically meaningful assumption of vanishing expansion scalar which describes the rate of change of small volumes of the fluid. The expansionfree fluid evolution should imply the formation of a vacuum cavity within spherically symmetric fluid distribution [22]. Taking $\Sigma^{(i)}$ (i stands for internal) to be the boundary surface of that vacuum cavity and matching this surface with Minkowski spacetime, we have

$$M(t, r) \stackrel{\Sigma^{(i)}}{=} 0, \quad -p_r - \frac{4}{3}\eta\sigma \stackrel{\Sigma^{(i)}}{=} \frac{T_{11}^{(D)}}{B^2} + \frac{T_{01}^{(D)}}{AB}. \quad (3.21)$$

The proper time and radial derivatives are given by

$$D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad D_C = \frac{1}{C'} \frac{\partial}{\partial r}, \quad (3.22)$$

where C is the areal radius of a spherical surface inside the boundary $\Sigma^{(e)}$, as measured from its area. The velocity of the collapsing fluid is defined by the proper time derivative of C , i.e.,

$$U = D_T C = \frac{\dot{C}}{A} \quad (3.23)$$

which is always negative. Using this expression, Eq.(3.17) implies that

$$E \equiv \frac{C'}{B} = [1 + U^2 + \frac{2M}{C}]^{1/2}. \quad (3.24)$$

When we make use of Eqs.(2.9), (2.10) and (3.22) in Eq.(2.12), we obtain

$$E \left[\frac{1}{3} D_C (\Theta - \sigma) - \frac{\sigma}{C} \right] = -\frac{4\pi}{F} \frac{T_{01}^{(D)}}{AB}. \quad (3.25)$$

The rate of change of mass in Eq.(3.17) with respect to proper time, with the use of Eqs.(2.11)-(2.14), is given by

$$D_T M = \frac{-4\pi}{F} \left[\left(p_r + \frac{4}{3} \eta \sigma + \frac{T_{11}^{(D)}}{B^2} \right) U - E \frac{T_{01}^{(D)}}{AB} \right] C^2. \quad (3.26)$$

This represents variation of total energy inside a collapsing surface of radius C . In the case of collapse $U < 0$, the terms inside the first round brackets increases the energy density through the rate of work being done by the effective radial pressure $p_r + \frac{4}{3} \eta \sigma$. The presence of dark fluid component shows the contribution of DE having large negative pressure. These terms appear with positive sign representing negative effect, hence decrease the rate of change of mass with respect to time. Now, it depends upon the strength of DE terms that they may balance the positive effect of effective radial pressure or overcome on them. Likewise, we have

$$D_C M = \frac{4\pi}{F} \left[\rho + \frac{T_{00}^{(D)}}{A^2} - \frac{U}{E} \frac{T_{01}^{(D)}}{AB} \right] C^2. \quad (3.27)$$

This equation describes how energy density and curvature terms influence the mass between neighboring surfaces of radius C in the fluid distribution.

Here the rate would decrease in the consecutive surfaces by the repulsive effect of DE. Integration of Eq.(3.27) with respect to "C" leads to

$$M = 4\pi \int_0^C \frac{C^2}{2F} \left[\rho + \frac{T_{00}^{(D)}}{A^2} - \frac{U}{E} \frac{T_{01}^{(D)}}{AB} \right] dC. \quad (3.28)$$

The dynamical equations can be obtained from the contracted Bianchi identities. Consider the following two equations

$$\left(T^{\alpha\beta} + T^{\alpha\beta (D)} \right)_{;\beta} u_\alpha = 0, \quad \left(T^{\alpha\beta} + T^{\alpha\beta (D)} \right)_{;\beta} \chi_\alpha = 0 \quad (3.29)$$

which yield

$$T^{\alpha\beta (D)}_{;\beta} u_\alpha = -\frac{1}{A} \left[\dot{\rho} + (\rho + p_r + \frac{4}{3}\eta\sigma) \frac{\dot{B}}{B} + 2(\rho + p_\perp - \frac{2}{3}\eta\sigma) \frac{\dot{C}}{C} \right], \quad (3.30)$$

$$\begin{aligned} T^{\alpha\beta (D)}_{;\beta} \chi_\alpha &= \frac{1}{B} \left[(p_r + \frac{4}{3}\eta\sigma)' + (\rho + p_r + \frac{4}{3}\eta\sigma) \frac{A'}{A} \right. \\ &\quad \left. + 2(p_r - p_\perp + 2\eta\sigma) \frac{C'}{C} \right]. \end{aligned} \quad (3.31)$$

Using Eqs.(2.9), (2.10), (3.22) and (3.24), it follows that

$$T^{\alpha\beta (D)}_{;\beta} u_\alpha = -[D_T \rho + \frac{1}{3}(3\rho + p_r + 2p_\perp)\Theta + \frac{2}{3}(p_r - p_\perp - 2\eta\sigma)\sigma], \quad (3.32)$$

$$T^{\alpha\beta (D)}_{;\beta} \chi_\alpha = ED_C(p_r + \frac{4}{3}\eta\sigma) + (\rho + p_r + \frac{4}{3}\eta\sigma)a + 2(p_r - p_\perp + 2\eta\sigma) \frac{E}{C}. \quad (3.33)$$

The acceleration $D_T U$ of the collapsing matter inside the hypersurface is obtained by using Eqs.(2.13), (3.22) and (3.24) as follows

$$D_T U = -\frac{M}{C^2} - \frac{4\pi}{F} \left(p_r + \frac{4}{3}\eta\sigma + \frac{T_{11}^{(D)}}{B^2} \right) C + Ea. \quad (3.34)$$

Substituting a from Eq.(3.34) into (3.33), it follows that

$$\begin{aligned}
& (\rho + p_r + \frac{4}{3}\eta\sigma)D_T U \\
= & -(\rho + p_r + \frac{4}{3}\eta\sigma) \left[\frac{M}{C^2} + \frac{4\pi}{F} \left(p_r + \frac{4}{3}\eta\sigma + \frac{T_{11}^{(D)}}{B^2} \right) C \right] \\
& - E^2 \left[D_C(p_r + \frac{4}{3}\eta\sigma) + \frac{2}{C}(p_r - p_\perp + 2\eta\sigma) \right] \tag{3.35}
\end{aligned}$$

This equation shows the role of different forces on the collapsing process. The factor within the brackets on the left hand side stands for "effective" inertial mass and the remaining term is acceleration. The first term on the right hand side represents gravitational force. The term within the first square brackets shows how shear viscosity and DE affect the passive gravitational mass. The first two terms in the second square brackets are gradient of the effective pressure and effect of local anisotropy of pressure with negative sign which increases the rate of collapse.

4 Expansionfree Evolution of Self-gravitating Fluid

In this section, we take $\Theta = 0$ and discuss expansionfree evolution of the self-gravitating fluid. In such a case, Eq.(2.10) yields

$$\frac{\dot{B}}{B} = -2\frac{\dot{C}}{C} \quad \Rightarrow \quad B = \frac{g_1(r)}{C^2}, \tag{4.1}$$

where $g_1(r)$ is an arbitrary function. Implication of vanishing Θ requires that innermost shell of the fluid should be away from the center of the collapsing sphere. This situation initiates the formation of a vacuum cavity at the center [22]. The physical meaning of this condition is explained by taking two different definitions of radial velocity of the fluid. Substituting Eq.(4.1) and value of $T_{01}^{(D)}$ from Eq.(2.15) in (2.12), we get

$$2 \left(\frac{\dot{C}'}{\dot{C}} - \frac{A'}{A} - 2\frac{C'}{C} \right) + \left(\frac{\dot{F}'}{F} - \frac{A'\dot{F}}{AF} \right) \frac{C}{\dot{C}} + 2\frac{F'}{F} = 0. \tag{4.2}$$

Integration yields

$$A = \frac{F^2 \dot{C} C^2}{\tau(t)} e^{-\int (\frac{\dot{F}'}{F} - \frac{A' \dot{F}}{A F}) \frac{C}{\dot{C}} dr}. \quad (4.3)$$

Thus the interior metric becomes

$$ds^2 = \left(\frac{F^2 \dot{C} C^2}{\tau_1(t)} \exp \left[- \int \left(\frac{\dot{F}'}{F} - \frac{A' \dot{F}}{A F} \right) \frac{C}{\dot{C}} dr \right] \right)^2 dt^2 - \left(\frac{g_1}{C^2} \right)^2 dr^2 - C^2(t, r) (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.4)$$

The above metric represents spherically symmetric anisotropic fluid which is going under shearing expansionfree evolution.

To proceed further, it is necessary to assume the condition of constant scalar curvature ($R = R_c$), according to which $F(R_c) = \text{constant}$. In $f(R)$ theory, the case of constant scalar curvature exhibits behavior just like solutions with cosmological constant in GR. This is one of the reason why the DE issue can be addressed by using this theory. In this case metric reduce to the following

$$ds^2 = \left(\frac{F_c^2 \dot{C} C^2}{\tau_1(t)} \right)^2 dt^2 - \frac{1}{C^4} dr^2 - C^2(t, r) (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4.5)$$

Here we have chosen $g_1(r) = 1$ without loss of generality. For this metric, the scalar curvature (2.16) becomes

$$\begin{aligned} R = & 2 \left[\left(\frac{\dot{C}'' C^4}{\dot{C}} + 2C'' C^3 + \frac{4\dot{C}'' C' C^3}{\dot{C}} + 2C''^2 C^2 + \frac{2\dot{C}'' C' C^3}{\dot{C}} + 9C''^2 C^2 \right. \right. \\ & + 4\dot{C}^2 C^2 + 2\ddot{C} C^3 - \frac{2\dot{\tau} \dot{C} C^3}{\tau} + 2C''' C^3 + \frac{1}{C^2} \Big) + \frac{\tau_1^2}{F_c^4} \left(\frac{\ddot{C}}{C} - \frac{6\dot{C}^2}{C^2} \right. \\ & \left. \left. - \frac{2\ddot{C}}{\dot{C}^2 C^5} + \frac{2\dot{\tau}}{\tau C^5 \dot{C}} - \frac{2\ddot{C}}{\dot{C} C^3} + \frac{7}{C^6} \right) \right]. \end{aligned} \quad (4.6)$$

One can find such values of C and τ_1 for which the condition of constant scalar curvature is satisfied. For constant scalar curvature, the components of dark fluid (2.15) reduce to

$$T_{00}^{(D_c)} = \frac{A^2 \Omega}{8\pi}, \quad T_{01}^{(D_c)} = 0, \quad T_{11}^{(D_c)} = T_{22}^{(D_c)} = \frac{-C^2 \Omega}{8\pi}, \quad (4.7)$$

where

$$\Omega = \frac{f(R_c) + R_c F(R_c)}{2}. \quad (4.8)$$

The corresponding junction conditions (3.18), (3.20) and (3.21) will take the form

$$\begin{aligned} M(t, r) &\stackrel{\Sigma^{(e)}}{=} m & p_r &\stackrel{\Sigma^{(e)}}{=} \frac{\Omega}{8\pi}, \\ M(t, r) &\stackrel{\Sigma^{(i)}}{=} 0, & p_r &\stackrel{\Sigma^{(i)}}{=} \frac{\Omega}{8\pi}. \end{aligned} \quad (4.9)$$

Using Eq.(4.5) in the field equations (2.11)-(2.14), it follows that

$$-2C^3 C''' - 5C^2 C'^2 + \frac{1}{C^2} - 3 \frac{\tau_1^2}{F_c^4 C^6} = \frac{1}{F_c} (8\pi\rho + \Omega), \quad (4.10)$$

$$\frac{1}{3} D_C \sigma + \frac{\sigma}{C} = 0, \quad (4.11)$$

$$\begin{aligned} &\frac{\tau_1^2}{F_c^4 \dot{C} C^5} \left(3 \frac{\dot{C}}{C} - 2 \frac{\dot{\tau}_1}{\tau_1} \right) + C^3 C' \left(2 \frac{\dot{C}'}{\dot{C}} + 5 \frac{C'}{C} \right) - \frac{1}{C^2} \\ &= \frac{1}{F_c} (8\pi p_r - \Omega), \end{aligned} \quad (4.12)$$

$$\begin{aligned} &\frac{-\tau_1^2}{F_c^4 \dot{C} C^5} \left(6 \frac{\dot{C}}{C} - \frac{\dot{\tau}_1}{\tau_1} \right) - C^4 \left(\frac{\dot{C}''}{\dot{C}} + 7 \frac{\dot{C}' C'}{\dot{C} C} + 3 \frac{C''}{C} + 10 \frac{C'^2}{C^2} \right) \\ &= \frac{1}{F_c} (8\pi p_\perp - \Omega). \end{aligned} \quad (4.13)$$

Similarly, Eqs.(3.26)-(3.27) become

$$D_T M = \frac{-4\pi U}{F_c} \left(p_r - \frac{\Omega}{8\pi} \right) C^2, \quad (4.14)$$

$$D_C M = \frac{4\pi}{F_c} \left(\rho + \frac{\Omega}{8\pi} \right) C^2 \quad (4.15)$$

implying that

$$M = \frac{4\pi}{F_c} \int_0^C \left(\rho + \frac{\Omega}{8\pi} \right) C^2 dC. \quad (4.16)$$

Further, Eq.(4.14) yields

$$8\pi p_r = \frac{-2\dot{M}F_c}{C^2 \dot{C}} + \Omega. \quad (4.17)$$

For Schwarzschild mass m , Eq.(4.17) is fully consistent with the junction conditions Eq.(4.9).

On the similar conditions, Bianchi identities are reduced to the following

$$\dot{\rho} + 2(p_{\perp} - p_r)\frac{\dot{C}}{C} + \frac{\Omega}{16\pi} \frac{\tau_1(2C\dot{C}^2 + C^2\ddot{C}) - \dot{\tau}_1\dot{C}C^2}{\tau_1 C^2 \dot{C}} = 0, \quad (4.18)$$

$$p'_r + (\rho + p_r)\frac{\dot{C}'}{\dot{C}} + 2(\rho + 2p_r - p_{\perp})\frac{C'}{C} - \frac{\Omega}{\pi} \frac{C'}{C} = 0. \quad (4.19)$$

We would like to mention here that in GR for isotropic fluid, i.e., $p_r = p_{\perp}$, Eq.(4.18) implies that energy density ρ depends only on r . However, in $f(R)$ theory it remains the function of both time and radial coordinate. Integration of Eq.(4.10) after some manipulation yields

$$C'^2 = \frac{1}{C^4} + \frac{\tau_2 - 2m}{C^5} + \frac{\tau_1^2}{F_c^4} \frac{1}{C^8} - \frac{\Omega}{C^2} \left(\frac{1}{6\pi} - \frac{1}{F_c} \right). \quad (4.20)$$

Here $\tau_2(t)$ is an arbitrary integration function. Using Eqs.(4.5) and (4.20) into Eq.(3.17), we obtain

$$\tau_2 = -\frac{\Omega}{k} \left(\frac{1}{6\pi} - \frac{1}{F_c} \right). \quad (4.21)$$

5 Skripkin Model

In the Skripkin model, it is assumed that fluid has isotropic pressure ($p_r = p_{\perp} = p$) and constant energy density, i.e., $\rho = \rho_0$. Here the fluid is assumed to be at rest initially and after that there is a sudden explosion at the center keeping the Skripkin condition. It is noted here that in GR, the expansion scalar automatically vanishes for the Skripkin conditions by virtue of Eq.(3.32). However, in $f(R)$ gravity, it will not vanish and we take expansionfree evolution in order to study Skripkin model. Consequently, Eq.(4.20) becomes

$$C'^2 = \frac{1}{C^4} + \frac{\tau_2}{C^5} + \frac{\tau_1}{F_c^4 C^8} - \frac{k}{C^2} - \frac{\Omega}{F_c} \frac{1}{C^2}, \quad (5.22)$$

where

$$k = \frac{8\pi\rho_0 F_c}{3}. \quad (5.23)$$

Under Skripkin conditions, Eq.(3.35) takes the form

$$(\rho_0 + p)D_T U = -(\rho_0 + p) \left[\frac{m}{C^2} + 4\pi p C - \frac{\Omega C}{2} \right] - E^2 D_C p, \quad (5.24)$$

Using Eq.(5.22) in Eq.(4.12), we obtain

$$\frac{8\pi p}{F_c} = - \left(3k + 2 \frac{\Omega}{F_c} \right) + \frac{\dot{\tau}_2}{C^2 \dot{C}}. \quad (5.25)$$

For isotropic pressure, Eq.(4.9) becomes

$$p \stackrel{\Sigma^{(e)}}{=} \frac{\Omega}{8\pi}. \quad (5.26)$$

Substituting Eq.(5.26) in Eq.(5.25) and integrating, we get

$$\tau_2 = \left(k + \frac{\Omega}{3F_c} \right) C_{\Sigma^{(e)}}^3 + c_1, \quad (5.27)$$

where c_1 is an integration constant. Using Eqs.(4.5) and (5.22) into Eq.(3.17), the mass function becomes

$$M = \frac{k}{2}(C^3 - C_{\Sigma^{(e)}}^3) + \frac{\Omega}{2F_c}(C^3 - \frac{1}{3}C_{\Sigma^{(e)}}^3) - \frac{c_1}{2}. \quad (5.28)$$

The total mass of the configuration m is obtained by measuring M on the boundary $\Sigma^{(e)}$ as follows

$$M = \frac{\Omega}{3F_c} C_{\Sigma^{(e)}}^3 - \frac{c_1}{2} = m, \quad (5.29)$$

Substituting value of c_1 from Eq.(5.29) into Eqs.(5.27) and (5.28), we have

$$\tau_2 = \left(k + \frac{\Omega}{F_c} \right) C_{\Sigma^{(e)}}^3 - 2m, \quad (5.30)$$

$$M = \frac{1}{2} \left(k + \frac{\Omega}{F_c} \right) (C^3 - C_{\Sigma^{(e)}}^3) + m, \quad (5.31)$$

Applying the matching conditions (3.21) on the boundary surface of the cavity inside the fluid, Eq.(5.31) yields

$$m = \frac{1}{2} \left(k + \frac{\Omega}{F_c} \right) (C_{\Sigma^{(e)}}^3 - C_{\Sigma^{(i)}}^3). \quad (5.32)$$

Using this equation in Eq.(5.30), we obtain

$$\tau_2 = \left(k + \frac{\Omega}{F_c} \right) C_{\Sigma^{(i)}}^3. \quad (5.33)$$

Differentiating Eq.(5.32) with respect to "t", we have

$$C_{\Sigma^{(e)}}^2 \dot{C}_{\Sigma^{(e)}} = C_{\Sigma^{(i)}}^2 \dot{C}_{\Sigma^{(i)}} \quad (5.34)$$

implying that

$$A_{\Sigma^{(e)}} = A_{\Sigma^{(i)}} \quad \text{and} \quad U_{\Sigma^{(e)}} = \frac{C_{\Sigma^{(i)}}^2}{C_{\Sigma^{(e)}}^2} U_{\Sigma^{(i)}}. \quad (5.35)$$

This result can also be obtained by definition of U on the boundary surfaces.

Applying Skripkin condition of constant energy density, Eq.(4.16) yields

$$M = \frac{4\pi}{F_c} \left(\rho + \frac{\Omega}{8\pi} \right) \frac{C^3}{3}. \quad (5.36)$$

Differentiating with respect to "t", we have

$$\dot{M} = \frac{4\pi}{F_c} \left(\rho + \frac{\Omega}{8\pi} \right) C^2 \dot{C}. \quad (5.37)$$

Substitution of Eq.(5.37) into Eq.(4.14) with isotropic pressure yields

$$p = -\frac{\rho_0}{F_c} + \frac{\Omega}{8\pi} \left(1 - \frac{1}{F_c} \right) = \text{constant}. \quad (5.38)$$

Comparing Eq.(5.38) with Eq.(5.26), we obtain

$$\rho_0 \stackrel{\Sigma^{(e)}}{=} \frac{\Omega}{8\pi}. \quad (5.39)$$

This shows that Skripkin model is eliminated by the junction conditions. It is mentioned here that in GR, $\rho_0 = 0$ while in our case, it turns out to be a non-zero constant.

Since pressure will be maximum on some spherical surface $C = C_s$ (say), hence pressure gradient must vanish on this surface. Thus it follows from Eq.(4.19) that

$$(\rho + p)(C^2 C')' - \frac{\Omega}{32\pi} \frac{C}{C'} \stackrel{s}{=} 0. \quad (5.40)$$

The solution of this equation yields the surface C_s which divides the fluid into two regions with a positive and negative pressure gradients respectively in the inner and outer sides of the surface.

6 Summary and Conclusions

The most important feature of $f(R)$ gravity is concerned with the issue of DE. In this paper, we have studied the effects of $f(R)$ DE on the dynamics of a self-gravitating spherically symmetric star. The star is made up of viscous anisotropic fluid distribution and dissipating energy in the form of shearing viscosity. This work extends and generalizes our recent paper [28]. As a special case, the Skripkin model is studied which is based on the simplest conditions of isotropic pressure and constant energy density.

This study is devoted to the expansionfree evolution of the fluid collapse resulting in the formation of a vacuum cavity within the fluid distribution. We have developed junction conditions on two hypersurfaces. One is the $\Sigma^{(e)}$, separating the fluid distribution from the Schwarzschild spacetime while the other is $\Sigma^{(i)}$, the boundary of internal cavity within which we have Minkowski spacetime.

It is found that DE arising from the curvature fluid affects the whole dynamics of the gravitational collapse due to its repulsive effect. For example, it decreases the rate of change of mass of collapsing sphere with respect to time and adjacent surfaces. The comparison with the corresponding GR yields more general results and hence provides extra degree of freedom. For example, in Skripkin model, the comparison with [22],[23] is given as follows

- In GR, implication of Skripkin conditions on the dynamical equations yields expansionfree evolution, i.e., $\Theta = 0$. However, no such relation is obtained in $f(R)$ gravity.
- In GR, isotropic pressure with nondissipation leaves the energy density as only a function of radial coordinate. Here, Eq.(4.18) does not imply that energy density is time independent due to presence of curvature fluid (even in the simplest case of constant scalar curvature).
- Matching of results on the boundary surface $\Sigma^{(e)}$ gives a non-zero constant value (based on constant scalar curvature) to Skripkin energy density ρ_0 while it becomes zero in GR.

It is worth mentioning here that Skripkin model is completely consistent with the junction conditions. Moreover, the expansionfree models with the appearance of a vacuum cavity may be used to study the formation of voids at cosmological scales [30] for any type of fluid.

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